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Finding Formulas for the Number of Pieces Resulting from k (n-1)-Dimensional Linear Cuts on a Unit Ball in Euclidean n-Space

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Our research question examined the maximum number of pieces into which a convex region can be cut using k number of (n-1)-dimensional linear cuts. This problem is related to many other mathematical fields, like topology and geometry, and it also has connections to the medical field. A CT scan uses combinations of many X-ray images at different angles to produce cross-sectional images, which is exactly what this research problem examined in the different convex regions. Specifically, we studied the regions by analysing our drawings and formulating proofs.
We were able to find some of the formulas for our problem for k-dimensional convex figures. By using the summation formulas: ∑(_i=1^(n   )) i^k , we have found explicit formulas up to i=6 for g_i (n), where g_i (n) is the number of regions into which each figure is cut and i is the dimension number. We have also found a recursive formula for i-dimensions:

$$g_i(n) = g_i(n-1) + g_{i-1}(n-1).$$

We have also worked on 2-dimensional convex shapes with convex regions removed from it. This problem can be translated into more dimensions with more regions removed. We have found a recursive formula for this region if one straight cut can pass through all the holes, assuming the holes are in a straight line:

$$f_i(n) = (n-1) + f_{i-1}(n),$$

where f_i (n) is the number of regions into which the shape is cut, i is the number of holes inside of the shape, and n is the number of straight cuts. We have also found that if these holes are not in a straight line, then the number of pieces into which the region can be cut is the same as the largest number of holes that are in a straight line.
This research has connections to the medical field and could help doctors better understand the mathematical principles behind the machines they use. Further research could investigate the formula for finding the number of holes into which higher dimensional regions can be cut. We can also continue research directed at finding a general, non-recursive formula for the number of pieces into which each region can be cut.

Statement of Research Advisor:

For the first result of our research, we used an inductive technique to obtain the formulas. Although the solution was known to me, a different technique was used by Elizabeth. The problem was then generalized to a convex body with a convex hole or a set of holes of different configurations. Elizabeth used different techniques to obtain some new partial solutions to this generalization. Additional properties of the counting functions were derived. For example, the leading coefficient was calculated exactly without induction; this was also new to both researchers—Michel Smith, Mathematics.