Variations of Toeplitz' Conjecture

By:Doyon Kim, Andras Bezdek

In 1911, Otto Toeplitz conjectured the following: Every Jordan curve in the plane contains all four vertices of a square. In mathematics, a curve is called Jordan curve if it is planar (can be drawn on a paper), simple (does not cross itself) and closed (walking along the curve starting at any point of the curve one will come back to the starting point). We say that a polygon P is inscribed in a curve C if all vertices of P are on C. So far Toeplitz' conjecture is solved for curves that are "smooth enough," but the problem in its full generality is still open. For instance, all curves people can draw with a pencil on a piece of paper, such as circles and polygons, are smooth, and the conjecture is solved for these cases. A closer look at Jordan curves, however, reveals that they can be as complicated as fractals. This complexity renders the problem difficult in spite of its intuitive nature. The general question addressed in this study can be stated as, "Under what condition does a Jordan curve have a specific inscribed polygon?"

We considered the following new variation of the original problem. We said that a polygon P is strongly inscribed in a curve C, if P is inscribed in C and if its interior is contained by the region enclosed by the Jordan curve. We wanted to find a general statement regarding the existence of a polygon P strongly inscribed in a curve C. Even though much research has focused on the original Toeplitz' conjecture, nobody has researched the problem with this new condition imposed. This condition leads to complications, mainly because it prevents us from using a standard "continuity argument," where one continuously changes both the size and the position of a strategically selected square and argues that along this change, a desired square must appear at least once.

The following are the theorems we found:

- Theorem 1: For every , there is a Jordan curve that does not have a strongly inscribed triangle whose smallest angle.
- Theorem 2: Every Jordan curve has a strongly inscribed triangle.
- Theorem 3: Let C be a Jordan curve and let T be a triangle. Then there exists a triangle T´ similar to T such that the interior of T´ is contained by the region enclosed by C and two of the vertices of T´ belong to C.
- Theorem 4: Let C be a Jordan curve and let D be a quadrilateral. Then there is a quadrilateral D´ similar to D such that the interior of D´ is contained by the region enclosed by C and two of the vertices of D´ belong to C.

We were able to develop a new geometric method to prove our theorems. Next we will try to generalize Theorems 3 and 4 for all convex polygons. Since we succeeded in proving any Jordan curve contains at least two vertices of a triangle or a quadrilateral of any shape, we expect to prove the same statement for convex polygons with more than four 4 vertices. We also plan to identify classes of Jordan curves for which you can find strongly inscribed triangles of any prescribed shape.

Statement of Research Advisor:

Discrete geometry is rich in fascinating, simple-stated problems. Doyon Kim worked on new variants of a beautiful, over 100-year-old problem. The questions he was asked to answer gave him an opportunity to show that he is able to do independent research and prove new results. His research will be published in a peer-reviewed math journal.

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